Comment: The algebraic continuity theorem uses the algebraic limit theorem and the definition of continuous:

Algebraic Continuity Theorem:

1. The sum of two continuous fcns is continuous.
2. " difference " " " " "
3. " product " " " "
4. " quotient " " " " " " ".
as long as the denominator is nonzero.
5. " composition " " " " "

As a result, linear fcns are continuous,
polynomials " "
rational fcns are continuous on their domain
fcns defined by convergent Taylor series are continuous.

Two important facts

1. The continuous image of a compact set is compact.

i.e. If $f : U \to V$ is continuous, with $U, V \subseteq \mathbb{C}$, and if $S \subseteq U$ is a compact subset of $\mathbb{C}$, then $f(S)$ is compact. $[f(S) = \{f(z) : z \in S\}.]$

[recall: compact $\Longleftrightarrow$ closed & bounded.].

② The continuous image of a connected set is connected

i.e. If $f: U \to V$ is continuous, with $U, V \subseteq \mathbb{C}$, and if $T \subseteq U$ is a connected subset of $\mathbb{C}$, then $f(T)$ is connected.

Important Example: The stereographic projection.

This is a function from the unit sphere in $\mathbb{R}^3$ whose image is $\mathbb{C}$ (really, $\mathbb{C} \cup \{\infty\}$).

boundary only no inside.

It is a 1-1 and onto continuous map that is actually conformal — it preserves angles between curves.

[unit sphere in $\mathbb{R}^3$ = $\{(a,b,c) : a^2+b^2+c^2=1\}$]
(centered at $(0,0,0)$, radius 1.)

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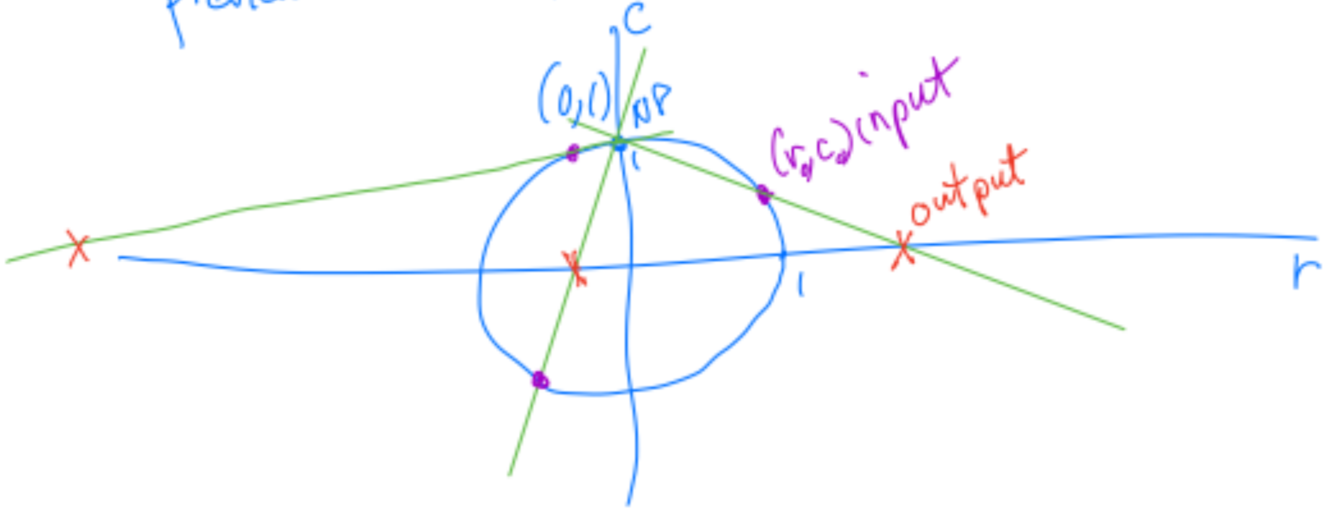
preview: stereographic projection on $\mathbb{R}^2 \to \mathbb{R}$



1

Given (r_0,c_0) on the unit circle $r^2+c^2=1$, $(r_0,c_0)\neq(0,1)$

I will make the line connecting $(0,1)$ and (r,c), and my map will be $\phi(r_0,c_0)$ = point on the r axis where that line hits. $\phi:\begin{pmatrix}\text{unit}\\\text{circle}\end{pmatrix}\to\underbrace{\mathbb{R}\cup\{\infty\}}_{\text{circle}}$

Let's compute this map:

slope of line $=\dfrac{c_0-1}{r_0}$, y-intercept $=1$

line equation: $c=\left(\dfrac{c_0-1}{r_0}\right)r+1$

where does it hit the r-axis?

$$c=0=\left(\frac{c_0-1}{r_0}\right)r+1\Rightarrow$$

$$\left(\frac{c_0-1}{r_0}\right)r=-1\Rightarrow r=\frac{-r_0}{c_0-1}$$

$$\Rightarrow r=\frac{r_0}{1-c_0}$$

Stereographic projection map:

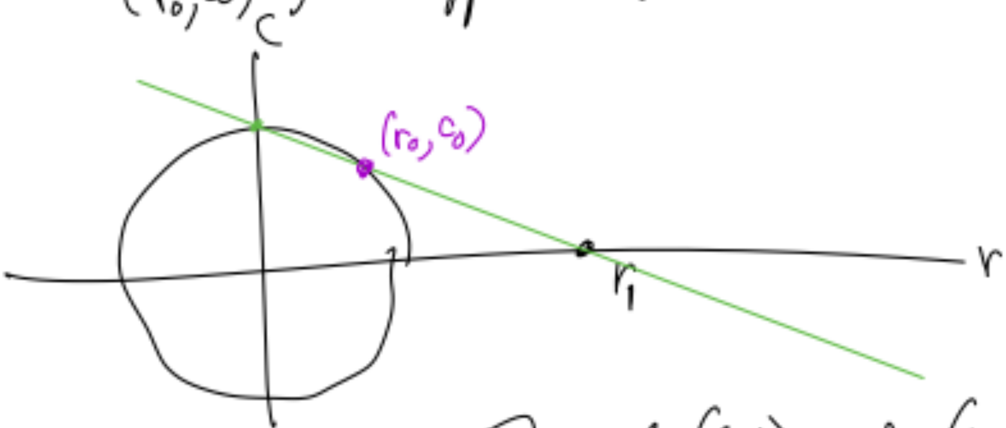
$$\boxed{\phi(r_0,c_0)=\frac{r_0}{1-c_0}}\ \text{if } (r_0,c_0)\neq(0,1)\quad \boxed{\phi(0,1)=\infty}$$

Notice: $\phi(r_0,0)=r_0$ (Stereographic projection maps "equator" to itself.)

ϕ maps (lower part of circle) to $[-1,1]$ on r-axis

ϕ maps (upper part of circle) to $[-\infty,-1)\cup(1,\infty]$.

inverse map: Given r on the r-axis, what point (r_0, c_0) gets mapped to it?



eqn of line: Through $(0,1)$ and $(r_1, 0)$.

$$\text{slope} = \frac{1}{-r_1} = \frac{-1}{r_1}.$$
$$y\text{-int} = 1$$

$$c = \left(\frac{-1}{r_1}\right) r + 1$$

$$r^2 + c^2 = 1$$

$$r^2 + \left(\frac{-r}{r_1} + 1\right)^2 = 1$$

$$r^2 + \frac{r^2}{r_1^2} + 1 - \frac{2r}{r_1} = 1$$

$$\left(1 + \frac{1}{r_1^2}\right) r^2 - \frac{2}{r_1} r = 0$$

$$r\left(\left(1 + \frac{1}{r_1^2}\right) r - \frac{2}{r_1}\right) = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{+2}{r_1\left(1 + \frac{1}{r_1^2}\right)} = \frac{+2r_1}{r_1^2 + 1}$$

$$C = \left(-\frac{1}{r_1}\right)(r) + 1$$

$$C = -\frac{1}{r_1}\left(\frac{2r_1}{r_1^2+1}\right) + 1 = -\frac{2}{r_1^2+1} + 1$$

$$= \frac{-2}{r_1^2+1} + \frac{r_1^2+1}{r_1^2+1} = \frac{r_1^2-1}{r_1^2+1}$$

$$\therefore (r_0, c_0) = \left(\frac{2r_1}{r_1^2+1}, \frac{r_1^2-1}{r_1^2+1}\right)$$

$$\phi^{-1}(r_1) = \left(\underset{r_0}{\frac{2r_1}{r_1^2+1}}, \underset{c_0}{\frac{r_1^2-1}{r_1^2+1}}\right)$$

we can check $r_0^2 + c_0^2 = 1$

$$\phi^{-1}(\infty) = (0,1)$$

Summary:

(r_0, c_0)
$r_0^2 + c_0^2 = 1$

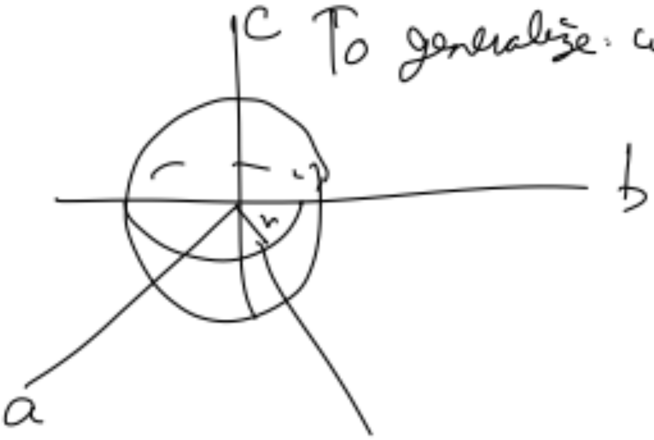
$$\phi(r_0, c_0) = \frac{r_0}{1-c_0} = r_1 \qquad \phi(0,1) = \infty$$

$$\phi^{-1}(r_1) = \left(\frac{2r_1}{r_1^2+1}, \frac{r_1^2-1}{r_1^2+1}\right) \qquad \phi^{-1}(\infty) = (0,1)$$

Comment: We could generalize to a map $\phi : S^n \to \mathbb{R}^n \cup \{\infty\}$

We'll stick to the case $n=2$ $\phi : S^2 \to \mathbb{R}^2 \cup \{\infty\}$
$= \mathbb{C} \cup \{\infty\}$

To generalize: we rotates around the c axis.



Think of r as the radius.

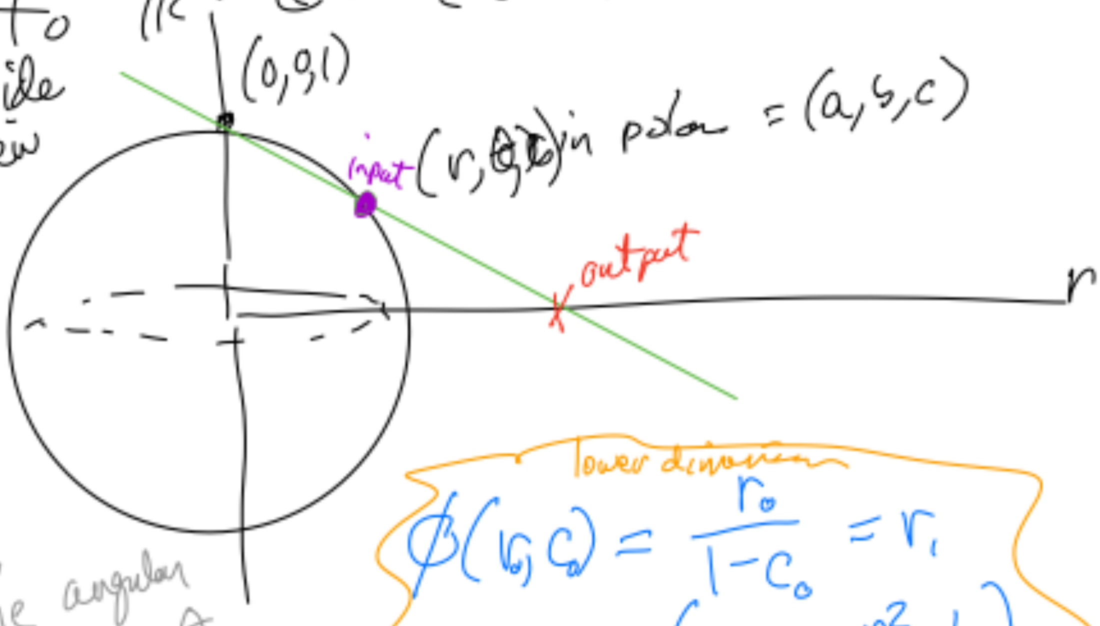
We will construct a map from

$S^2 \subseteq \mathbb{R}^3$

$S^2 = \{(a,b,c) : a^2+b^2+c^2=1\}$

to $\mathbb{R}^2 = \mathbb{C} = \{(x+iy)=(x,y) : x,y \in \mathbb{R}\}$.

side view



Lower dimension

$$\phi(r_0, c_0) = \frac{r_0}{1-c_0} = r_1$$

$$\phi^{-1}(r_1) = \left(\frac{2r_1}{r^2+1}, \frac{r_1^2-1}{r^2+1}\right)$$

The angular coordinate θ in the (a,b) plane does not change.

$$\phi(a_0, b_0, c_0) = (x, y)$$

$c_0 = c_0$
$a_0 = r_0 \cos\theta$
$b_0 = r_0 \sin\theta$

where $x = r_1 \cos\theta$
$y = r_1 \sin\theta$

$$\frac{\overbrace{r_0\cos\theta}^{a_0}}{1-c_0} = r_1\cos\theta = x$$

$$\frac{\overbrace{r_0\sin\theta}^{b_0}}{1-c_0} = r_1\sin\theta = y$$

Formula $\phi : S^2 \longleftrightarrow \mathbb{C} \cup \{\infty\}$

$$\boxed{\begin{aligned}\phi(a_0, b_0, c_0) &= \left(\frac{a_0}{1-c_0}, \frac{b_0}{1-c_0}\right) \\ &= \left(\frac{a_0}{1-c_0} + i\frac{b_0}{1-c_0}\right).\end{aligned}}$$

Inverse

$$\cancel{\phi^{-1}(r_1) = \left(\frac{\overset{r_0}{2r_1}}{r_1^2+1}, \frac{r_1^2-1}{r_1^2+1}\right)}$$

$$\phi^{-1}(\underbrace{r_1\cos\theta}_{x}, \underbrace{r_1\sin\theta}_{y}) = \left(\overbrace{\frac{2r_1\cos\theta}{r_1^2+1}}^{r_0\cos\theta}, \overbrace{\frac{2r_1\sin\theta}{r_1^2+1}}^{r_0\sin\theta}, \overbrace{\frac{r_1^2-1}{r_1^2+1}}^{c_0}\right)$$

$$\phi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

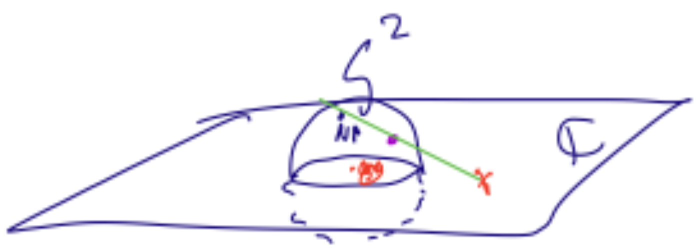
$r_1^2 = x^2+y^2.$

(Stereographic projection maps equator to itself.)

ϕ maps (lower part of sphere) to the unit disk in $\mathbb{C}$:–

ϕ maps (upper part of sphere) to the complement of the unit disk in $\mathbb{C}$.

Note: $S^2 \cap \mathbb{C}$ = unit circle in $\mathbb{C}$



Properties: Note: every plane intersects S^2 in a circle.

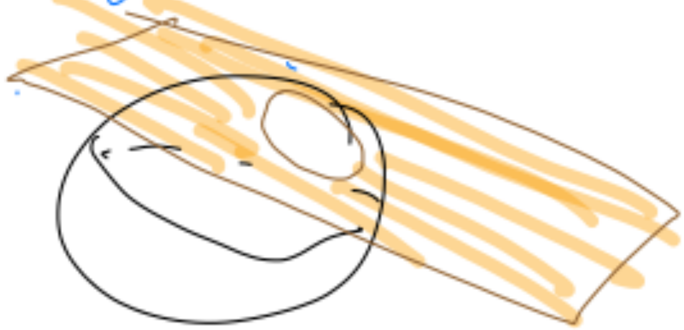
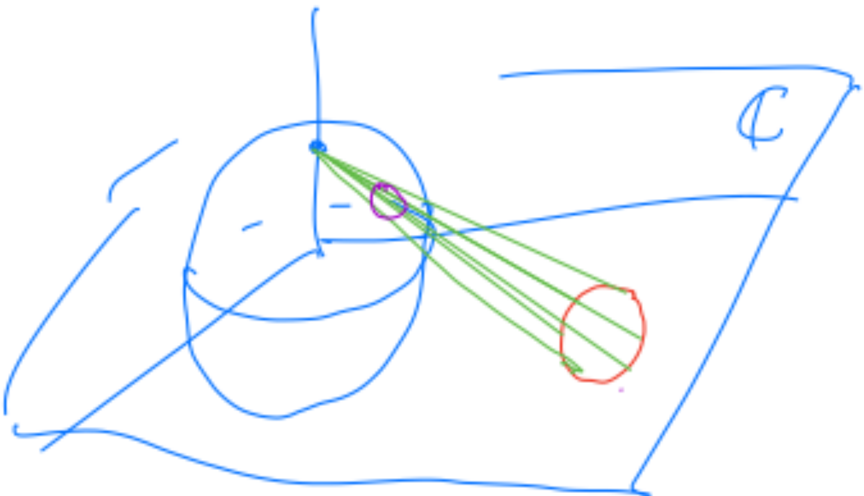
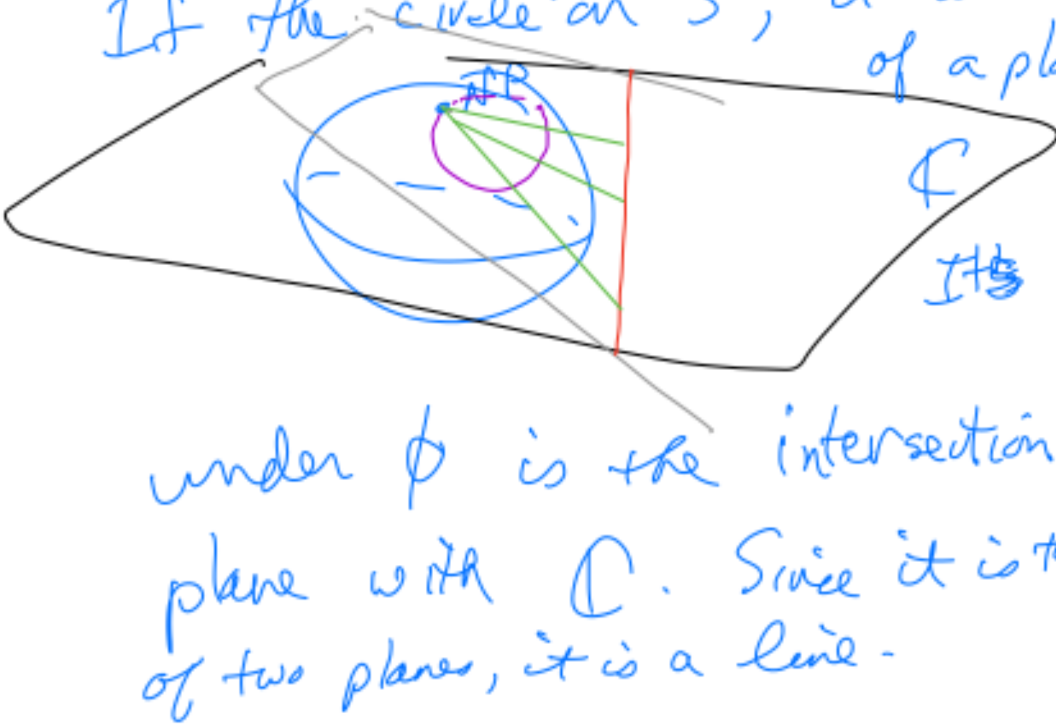


Image of every circle in S^2 under the stereographic projection map is either a circle or a line (you can prove this.)



If the circle is on S^2, it is the intersection of a plane through the north pole.



Its image under ϕ is the intersection of that plane with $\mathbb{C}$. Since it is the intersection of two planes, it is a line.